Derivative relation for thermopower in the quantum Hall regime

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Recently, Tieke *et al* (to be published in PRL) have observed the relation $S_{yx} = \alpha B \frac{dS_{xx}}{dB}$ for the components of the thermopower tensor in the quantum Hall regime, where α is a constant and B is the magnetic field. Simon and Halperin (PRL **73**, 3278 (1994)) have suggested that an analogous relation observed for the resistivity tensor $R_{xx} = \alpha B \frac{dR_{xy}}{dB}$ can be explained with a model of classical transport in an inhomogeneous medium where the local Hall resistivity is a function of position and the local dissipative resistivity is a small constant. In the present paper, we show that this new thermopower relation can be explained with a similar model. (This paper supersedes cond-mat/9705001 which was withdrawn)

1. Introduction

For a wide range of conditions, high mobility quantum Hall systems have been observed to display a derivative relation [1]

$$R_{xx} = \alpha_r B \frac{dR_{xy}}{dB} \tag{1}$$

where R_{xx} and R_{xy} are the diagonal and off diagonal components of the measured resistivity tensor $\vec{R}$, B is the magnetic field, and α_r is a sample dependent (and weakly temperature dependent) constant. In Ref. [2] an explanation for this relation was proposed based on a classical analysis of transport properties of a system with a local Hall resistivity $\rho_{xy}(\vec{r})$ that is a function of position and a local longitudinal resistivity ρ_{xx} which is a small constant. It was found that if the correlations in the disorder of $\rho_{xy}(\vec{r})$ exist on several length scales [3], then the derivative law can be reasonably explained. (A detailed review of Ref. [2] will be given in section 2 below).

In a recent letter, Tieke *et al.* [4] have observed an analogous derivative relation for the thermopower given by

$$S_{yx} = \alpha_s B \frac{dS_{xx}}{dB} \tag{2}$$

where S_{xx} and S_{yx} are the diagonal and off-diagonal parts of the thermopower tensor $\vec{S}$ and α_s is a constant found to be approximately equal to α_r. (The thermopower is defined via $\vec{E} = \vec{S}\vec{\nabla}T$ under conditions where no current is allowed to flow into or out of the sample with $\vec{E}$ the electric field and T the temperature). In Ref. [4], it was conjectured that similar physics may be at work in thermopower as for resistivity. The purpose of this paper is to demonstrate that the derivative relation for thermopower (Eq. 2) can be derived in a similar manner to the derivative relation for resistivity (Eq. 1).

2. Review of Resistivity Problem

We begin by reviewing the derivation of the derivative relation for resistivity (Eq. 1) that was proposed in Ref. [2]. In that work it is assumed that there is a local resistivity tensor $\rho(\vec{r})$ whose off diagonal component $\rho_{xy}(\vec{r}) = -\rho_{yx}(\vec{r})$ is some arbitrary function f of the local filling fraction $\nu(\vec{r})$

$$\rho_{xy}(\vec{r}) = f(\nu(\vec{r})) \tag{3}$$

with

$$\nu(\vec{r}) = n(\vec{r})\phi_0/B \tag{4}$$

where $n(\vec{r})$ is the local density and $\phi_0 = hc/e$ is the flux quantum. The density is assumed to have some average value $\langle n\rangle$ and some root mean square fluctuation δn, such that the filling fraction also has some average value $\langle\nu\rangle$, and some root means square fluctuation $\delta\nu$ given by

$$\langle\nu\rangle = \langle n\rangle\phi_0/B \quad ; \quad \delta\nu = \delta n\,\phi_0/B. \tag{5}$$

(Everywhere in this paper, $\langle\rangle$ is a spatial average, and δ is a root mean square fluctuation around this average). We will assume that the local fluctuations in density are smooth and are much smaller than the average density. Thus ρ_{xy} also has an average value $\langle\rho_{xy}\rangle$ and a fluctuation $\delta\rho_{xy} \ll \langle\rho_{xy}\rangle$ given by

$$\langle\rho_{xy}\rangle = f(\langle\nu\rangle) \quad ; \quad \delta\rho_{xy} = \delta\nu\,|f'(\langle\nu\rangle)|. \tag{6}$$

To complete the model, we must also include a mechanism for dissipation. We will consider a model discussed in Ref. [2] which assumes that the local dissipative resistivity $\rho_{xx} = \rho_{yy}$ is a small constant. The major result (Eq. 1) turns out to be relatively independent of the precise model of dissipation so long as the local dissipation is very small. For this resistive model, one must assume that $\rho_{xx} \ll \delta\rho_{xy}$.

In order to solve the transport problem we must satisfy current conservation, Maxwell's equation, and Ohm's law:

$$\vec{\nabla}\cdot\vec{j} = 0 \tag{7}$$

$$\vec{\nabla}\times\vec{E} = 0 \tag{8}$$

$$\vec{E} = \overleftrightarrow{\rho}\vec{j}. \tag{9}$$

These must be supplemented with the boundary condition that a fixed total current runs through the system, or equivalently, that the spatial average of the current $\langle \vec{j} \rangle$ has a specified value. Substituting Eq. 9 into Eq. 8 and using Eq. 7, we obtain the fundamental equation [2]

$$\vec{\nabla}\rho_{xy} \cdot \vec{j} - \rho_{xx}(\vec{\nabla} \times \vec{j}) = 0. \tag{10}$$

This equation determines the current paths through the system and hence the resistivity of the sample. There are two important things to note about this equation. To begin with, the solution to this equation is clearly independent of the average value of the Hall resistivity $\langle \rho_{xy} \rangle$ and can therefore only depend on its fluctuations. Since the current profile determines the dissipation, on dimensional grounds, in the limit of small ρ_{xx}, the macroscopic dissipative resistivity R_{xx} must scale as

$$R_{xx} = C_r(\delta\rho_{xy})^{1-\omega}\rho_{xx}^{\omega} \tag{11}$$

with C_r a dimensionless constant. We will find below that the exponent ω depends on the details of the disorder in the sample, but is typically small, and can often be quite close to zero. We will show below that a sufficiently small value of ω will allow us to derive the derivative relation for resistivity (Eq. 1). The important physical result in Eq. 11 is that the macroscopically measured dissipative resistivity can depend very strongly on the fluctuations in local ρ_{xy} and can be relatively independent of the microscopic dissipative resistivity ρ_{xx}.

The second thing to note about Eq. 10 is that for $\rho_{xx} = 0$, the current paths must flow perpendicular to the gradient of ρ_{xy}, or along an equi-ρ_{xy} contour. A nonzero ρ_{xx} in Eq. 10 can be viewed as a diffusion constant for the current distribution [5], and for sufficiently small ρ_{xx} the current cannot diffuse very far away from these contours. Thus, in the limit of small ρ_{xx}, in order for current to flow over distances large compared to the correlation length of the inhomogeneities (which is assumed to be small compared to the sample size), it must follow contours of ρ_{xy} that percolate across a macroscopic portion of the system. We know from percolation theory [2,5] that such a percolating contour will be extremely convoluted. Thus, for small ρ_{xx}, the current path is anomalously long so the macroscopic resistivity is anomalously large.

As we increase ρ_{xx} two things happen. On the one hand, the dissipation per unit length increases, but on the other hand the current can diffuse somewhat from the equi-ρ_{xy} contours cutting off corners of the long tortuous path, decreasing the length of the current path, and thus acting to decrease the dissipation. These competing effects keep the macroscopic dissipative resistivity R_{xx} relatively independent of the microscopic dissipative resistivity ρ_{xx}, thus keeping the exponent ω small. For Gaussian correlated disorder on a single length scale [2,5,6] it is found that $\omega = 3/13$. (For a similar model with viscous dissipation [2], one finds $\omega = 3/19$).

If disorder exists on several length scales, however, the exponent ω can be much smaller [2,5]. To see this we consider a system where there is Gaussian correlated disorder on two well separated length scales $l \ll l'$ which are both much less than the size of the system. Using the above argument we find that the dissipative resistivity ρ'_{xx} on a scale much larger than l but much less than l' would be $\rho'_{xx} \sim \rho_{xx}^{3/13}(\delta\rho_{xy})^{10/13}$. Now using ρ'_{xx} as a microscopic resistivity and repeating the argument for the disorder on length scale l' yields $R_{xx} \sim (\rho'_{xx})^{3/13}(\delta\rho_{xy})^{10/13} \sim \rho_{xx}^{9/169}(\delta\rho_{xy})^{160/169}$ or an exponent of $\omega = (3/13)^2$.

Throughout this work, we will assume that disorder exists on several length scales so that the exponent ω is very small. (The experimental observation of the derivative relation Eq. 1 for resistivity will be taken as one piece of evidence for disorder on several length scales. Further evidence is given in Ref. [3]).

We now show that a sufficiently small exponent ω results in the derivative relation shown in Eq. 1. Considering the case of $\omega = 0$, we have

$$R_{xx} = C_r\delta\rho_{xy} = C_r\delta\nu\,|f'(\langle\nu\rangle)| \tag{12}$$

Note that the macroscopic dissipative resistivity here depends entirely on the fluctuations in the microscopic ρ_{xy}. On the other hand, the macroscopic Hall resistivity is just

$$R_{xy} = \langle\rho_{xy}\rangle = f(\langle\nu\rangle) \tag{13}$$

Differentiation of this equation with respect to magnetic field (using $d\nu/dB = -\nu/B$) leads to Eq. 1 with $\alpha_r = C_r\delta n/\langle n\rangle$. In general, we do not know the value of C_r, but assuming it to be order unity yields α_r on the order of a few percent which is in agreement with experimental observation.

If the exponent ω is only slightly different from zero, then the resistivity law Eq. 1 will be observed to hold to a reasonably good approximation. If ω were substantially different from zero, one would have to know the precise dependence of ρ_{xx} on magnetic field to make any further statements.

3. Mapping Thermopower To Resistivity

In the case of thermopower, we will once again look for the effects of inhomogeneities in the local transport properties on the measured response of the sample. Thus, we consider a local [7] thermopower tensor $\overleftrightarrow{s}(\vec{r})$ such that

$$\vec{E} = \overleftrightarrow{\rho}\vec{j} + \overleftrightarrow{s}\vec{\nabla}T. \tag{14}$$

We will write $s_{xx}(\vec{r}) = s_{yy}(\vec{r})$ as a function g of the local filling fraction, and of the magnetic field B,

$$s_{xx}(\vec{r}) = g(\nu(\vec{r}), B) \equiv g_B(\nu(\vec{r})). \tag{15}$$

Note that, unlike for resistivity, we do not in general assume that s_{xx} is a function of ν only (this will be discussed further below). In microscopic derivations [8] of the thermopower tensor $\overleftrightarrow{s}$, appropriate for the samples studied in Ref. [4], it is found that the diagonal component s_{xx} is large compared to the off diagonal component

s_{yx} which is small or zero. Thus, we assume $s_{yx} = -s_{xy}$ is a small constant (which may be zero). Specifically, we will assume that $s_{yx} \ll \delta s_{xx}$. As in the case of the resistivity problem, the precise behavior of s_{yx} will not affect the outcome of the analysis so long as it remains very small. (In addition, our results do not depend on whether the thermopower is dominated by the "phonon-drag" or "diffusion" contributions [9]).

We would like to make the thermopower problem look more like the resistivity problem above. To do so, we define a fictitious current which 'flows' along the isothermal lines

$$\vec{j}^{\rm T} = \hat{z} \times \vec{\nabla} T \tag{16}$$

with $\hat{z}$ the unit vector normal to the plane. Since

$$\vec{\nabla} \cdot \vec{j}^{\rm T} = \vec{\nabla} \times \vec{\nabla} T = 0 \tag{17}$$

we have $\vec{j}^{\rm T}$ a conserved current analogous to the charge current $\vec{j}$ in the resistivity problem. In terms of this new current, Eq. 14 is written as

$$\vec{E} = \overleftrightarrow{\rho}\,\vec{j} + \overleftrightarrow{\rho}^{\rm T}\,\vec{j}^{\rm T} \tag{18}$$

where

$$\overleftrightarrow{\rho}^{\rm T}(\vec{r}) = \begin{pmatrix} \rho^{\rm T}_{xx} & \rho^{\rm T}_{xy}(\vec{r}) \\ -\rho^{\rm T}_{xy}(\vec{r}) & \rho^{\rm T}_{xx} \end{pmatrix} = \begin{pmatrix} s_{yx} & s_{xx}(\vec{r}) \\ -s_{xx}(\vec{r}) & s_{yx} \end{pmatrix} \tag{19}$$

with $\rho^{\rm T}_{xx} \ll \delta\rho^{\rm T}_{xy}$.

Thus, s_{yx} is mapped to a dissipative resistivity $\rho^{\rm T}_{xx}$ which is assumed to be a small constant, and s_{xx} is mapped to a Hall resistivity $\rho^{\rm T}_{xy}$ which is a function of the local filling fraction. This mapping is then quite suggestive that the thermopower law (Eq. 2) might be derived analogously to the resistivity law (Eq. 1).

4. Thermopower Law

In the case of thermopower for a quantum Hall sample, it is essential to realize that the lattice surrounding the two dimensional electron gas carries heat much more readily than the electrons (since the number of electrons in the layer is quite small). The lattice surrounding the two dimensional electron gas is assumed to be homogeneous so that when a thermal gradient is applied to the sample, $\vec{\nabla} T$ is completely uniform. (Note that this assumes good thermal equilibration between the lattice and the electrons.) Thus we should think of $\vec{\nabla} T$ (or equivalently $\vec{j}^{\rm T}$) as being applied externally to the sample and as being a fixed quantity which is spatially uniform. This is very different from the above electrical case where only the average value $\langle \vec{j} \rangle$ is fixed and the actual current distribution is quite inhomogeneous. Here, we must also demand that no net electrical current travels in the system ($\langle \vec{j} \rangle = 0$). By substituting Eq. 18 into Maxwell's equation (Eq. 8) and using current conservation (Eqs. 7 and 17) we obtain the fundamental equation

$$\vec{\nabla} \rho^{\rm T}_{xy} \cdot \vec{j}^{\rm T} + \vec{\nabla} \rho_{xy} \cdot \vec{j} - \rho_{xx} (\vec{\nabla} \times \vec{j}) = 0. \tag{20}$$

similar to Eq. 10. Recall here that both $\rho^{\rm T}_{xy}(\vec{r}) = s_{xx}(\vec{r}) = g_B(\nu(\vec{r}))$ and $\rho_{xy}(\vec{r}) = f(\nu(\vec{r}))$ are determined by the local filling fraction. Thus, their gradients are proportional via (see Eqs. 3, 15, and 19) $\vec{\nabla} \rho^{\rm T}_{xy} = \gamma \vec{\nabla} \rho_{xy}$ where

$$\gamma = \frac{g_B'(\langle \nu \rangle)}{f'(\langle \nu \rangle)}. \tag{21}$$

Similarly, we have $\delta\rho^{\rm T}_{xy} = \gamma \delta \rho_{xy}$. Note that here, as in elsewhere in this work, we have assumed that $\delta n / \langle n \rangle$ is small enough that we need only expand quantities linearly around the average density.

We can now define a new current

$$\vec{j}^{+} = \vec{j} + \gamma \vec{j}^{\rm T} \tag{22}$$

in terms of which the fundamental equation 20 can be rewritten as (recalling that $\vec{j}^{\rm T}$ is a constant)

$$\vec{\nabla} \rho_{xy} \cdot \vec{j}^{+} - \rho_{xx} (\vec{\nabla} \times \vec{j}^{+}) = 0 \tag{23}$$

which is precisely the same as Eq. 10. This must be supplemented by the boundary condition that $\langle \vec{j}^{+} \rangle = \langle \vec{j} \rangle + \gamma \langle \vec{j}^{\rm T} \rangle = \gamma \vec{j}^{\rm T}$.

We thus see that current $\vec{j}^{+}$ travels across the system in the same inhomogeneous percolative manner as the electrical current in the resistivity problem where $\vec{j}^{+}$ flows only through very narrow channels and is zero (or very small) throughout most of the volume of the system. Note that here $\vec{j}^{+}$ is made up of two pieces — a uniform piece $\vec{j}^{\rm T}$ which is nonzero, and an electrical piece $\vec{j}$ that is highly inhomogeneous but has a zero average and carries no net current. The two pieces are arranged to precisely cancel throughout most of the system and only leave a nonzero contribution to $\vec{j}^{+}$ in narrow channels.

Extending the analogy with the resistivity problem, we define a local 'electric' field

$$\vec{E}^{+} = \overleftrightarrow{\rho}\,\vec{j}^{+}, \tag{24}$$

in terms of the local electrical resistivity tensor. One can then calculate $\vec{E}^{+}$ precisely as described in section 2 and (so long as we assume disorder on several length scales) we obtain a macroscopic average of $\vec{E}^{+}$ that satisfies

$$\langle \vec{E}^{+} \rangle = \overleftrightarrow{R} \langle \vec{j}^{+} \rangle = \gamma \overleftrightarrow{R}\, \vec{j}^{\rm T} \tag{25}$$

with the components of $\overleftrightarrow{R}$ given by Eqs. 12 and 13.

We now calculate the actual physical electric field, by rewriting 18 as

$$\vec{E} = \vec{E}^{+} + \vec{E}^{-} \tag{26}$$

with

$$\vec{E}^{-} = \overleftrightarrow{\rho}^{-}\vec{j}^{\mathrm{T}} \tag{27}$$
$$\overleftrightarrow{\rho}^{-} = \overleftrightarrow{\rho}^{\mathrm{T}} - \gamma\overleftrightarrow{\rho}. \tag{28}$$

Note that $\delta\rho^{-}_{xy} = 0$ so $\overleftrightarrow{\rho}^{-}$ is a constant tensor and $\rho^{-}_{xx} \ll \gamma\delta\rho_{xy}$.

Now since $\vec{j}^{\mathrm{T}}$ and $\overleftrightarrow{\rho}^{-}$ are both uniform in space, Eq. 27 yields a $\vec{E}^{-}$ which is simply a constant. We can then write the macroscopic average of the physical electric field as

$$\langle\vec{E}\rangle = \overleftrightarrow{R}^{\mathrm{T}}\vec{j}^{\mathrm{T}} \tag{29}$$
$$\overleftrightarrow{R}^{\mathrm{T}} = (\gamma\overleftrightarrow{R} + \overleftrightarrow{\rho}^{-}). \tag{30}$$

For the diagonal component of $\overleftrightarrow{R}^{\mathrm{T}}$, we note that since $\rho^{-}_{xx} \ll \gamma\delta\rho_{xy}$, we have $\rho^{-}_{xx} \ll \gamma R_{xx}$ and we can neglect ρ^{-}_{xx} to write

$$\begin{aligned} R^{\mathrm{T}}_{xx} &= \gamma R_{xx} = C_r\gamma\delta\rho_{xy} \\ &= C_r\delta\rho^{\mathrm{T}}_{xy} = C_r\delta\nu\,|g'_B(\langle\nu\rangle)|. \end{aligned} \tag{31}$$

For the off diagonal component of $\overleftrightarrow{R}^{\mathrm{T}}$, on the other hand, we have

$$\begin{aligned} R^{\mathrm{T}}_{xy} &= \gamma R_{xy} + \langle\rho^{\mathrm{T}}_{xy} - \gamma\rho_{xy}\rangle \\ &= \langle\rho^{\mathrm{T}}_{xy}\rangle = g_B(\langle\nu\rangle). \end{aligned} \tag{32}$$

This result could also have been obtained by examining Eq. 18 and noting that $\langle\vec{j}\rangle$ is fixed to be zero.

We now convert $\overleftrightarrow{R}^{\mathrm{T}}$ back to a thermopower $\overleftrightarrow{S}$. Using the macroscopic version of Eq. 19 we obtain

$$S_{yx} = R^{\mathrm{T}}_{xx} = C_r\delta\rho^{\mathrm{T}}_{xy} = C_r\delta s_{xx} = C_r\delta\nu\,|g'_B(\langle\nu\rangle)| \tag{33}$$
$$S_{xx} = R^{\mathrm{T}}_{xy} = \langle\rho^{\mathrm{T}}_{xy}\rangle = \langle s_{xx}\rangle = g_B(\langle\nu\rangle). \tag{34}$$

which are the analogous results to Eqs. 12 and 13. Note that, just as R_{xx} in the resistivity problem is independent of the small value of the local ρ_{xx} and is determined by the spatial fluctuations in $\rho_{xy}(\vec{r})$, S_{yx} is independent of the small value of the local s_{xy} and is a reflection of local fluctuations in $s_{xx}(\vec{r})$. By differentiating Eq. 34 with respect to filling fraction and comparing with Eq. 33 we obtain

$$S_{yx} = -\alpha_s\nu\left.\frac{dS_{xx}}{d\nu}\right|_B \tag{35}$$

with $\alpha_s = \alpha_r = C_r\delta n/n$.

If $s_{xx} = g(\nu, B)$ were just a function of ν this would complete our derivation. However, in general, this is not the case. Thus, we write

$$B\frac{dS_{xx}}{dB} = -\nu\left.\frac{dS_{xx}}{d\nu}\right|_B + B\left.\frac{dS_{xx}}{dB}\right|_\nu , \tag{36}$$

and we now must assume that S_{xx} (or more specifically $g(\nu, B)$) varies strongly with ν at fixed B but only slowly with B at fixed ν. This is actually quite a reasonable expectation for any microscopic calculation [8], since S_{xx} oscillates quite strongly with ν. With this assumption, we can neglect the second term on the right of Eq. 36 to obtain the desired result

$$S_{yx} = \alpha_s B\,\frac{dS_{xx}}{dB} \tag{37}$$

One should note that this derivation leads to the result $\alpha_s = \alpha_r$ which is indeed observed experimentally [4].

5. Summary

In this work we use a model in which the local density determines the local Hall resistivity $\rho_{xy}(\vec{r})$ as well as the local diagonal thermopower $s_{xx}(\vec{r})$. The local dissipative resistivity ρ_{xx} as well as the off diagonal thermopower s_{yx} are assumed to be small constants such that $\rho_{xx} \ll \delta\rho_{xy}$ and $s_{yx} \ll \delta s_{xx}$. We also must assume that s_{xx} is a strong function of ν at fixed B but a weak function of B at fixed ν. Finally, assuming that the disorder has long range correlations, or exists on several different length scales such that the exponent ω is close to zero, we are able to derive the derivative relation for thermopower (Eq. 2), in close analogy with the derivation of the corresponding law for resistivity (Eq. 1).

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